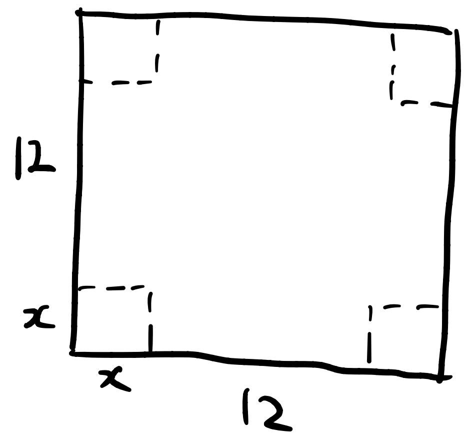
2C

3/8/25

1.



Domain of x: $0 < x < 6$

$$\ell = 12 - 2x = -2(x-6)$$

$$V = \ell^2 \times x$$

$$= 4x(x-6)^2$$

$$\frac{dV}{dx} = 4(x-6)^2 + 4x \cdot 2(x-6)(1)$$

$$= 4(x^2 - 12x + 36 + 2x^2 - 12x)$$

$$= 4(3x^2 - 24x + 36)$$

$$= 12(x^2 - 8x + 12)$$

$$= 12(x-6)(x-2)$$

At $\frac{dV}{dx} = 0$, $(x-6)(x-2) = 0$

$\therefore$ V has a single critical point at $x = 2$ as $x \neq 6$.

$$\lim_{x \to 6} V = 0$$

$$\lim_{x \to 0} V = 0$$

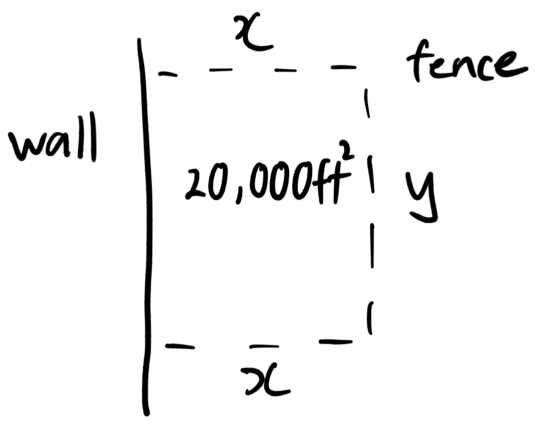
$f'(x) > 0$ when $x < 2$
$\Rightarrow$ V is increasing

$f'(x) < 0$ when $2 < x < 6$
$\Rightarrow$ V is decreasing

$\therefore$ $x = 2$ must be a maximum point.

$\therefore$ The corner squares of sides 2 inch or area 4 inch^2 maximizes the volume of the box.

2.



$$A = xy = 20000$$

$$\ell = 2x + y$$

$$\Rightarrow \ell = 2x + \frac{20000}{x}$$

$$\frac{d\ell}{dx} = 2 - \frac{20000}{x^2}$$

$$\frac{d\ell}{dx} = 0 \Rightarrow 2 - \frac{20000}{x^2} = 0$$

$$x^2 - 10000 = 0$$

$$x^2 = 10000$$

$$x = 100$$

As $x \to \infty$, $y \to 0 \Rightarrow \ell \to \infty$

$x \to 0$, $y \to \infty \Rightarrow \ell \to \infty$

$\therefore$ ℓ has one critical point at $x = 100$, and so it must be a minimum.

$\therefore$ $x = 100$, $y = 200$.

Shortest length of fence

$= 2(100) + 200$

$= 400\text{ft}$

4.

$$4x + l = 108$$

$$V = x^2 l$$

$$\Rightarrow V = x^2(108 - 4x)$$
$$= -4x^3 + 108x^2$$

$$\frac{dV}{dx} = -12x^2 + 216x$$

$$\frac{dV}{dx} = 0 \Rightarrow -12x(x - 18) = 0$$

$\therefore x = 0, 18$ are critical points.

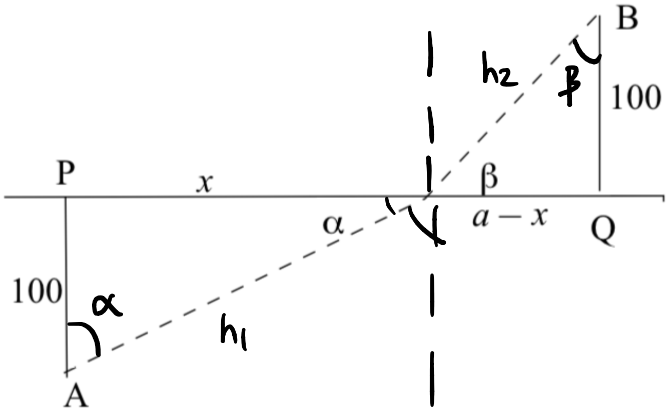
$\frac{dV}{dx} < 0 \Rightarrow V$ is decreasing when $x > 18$

$\frac{dV}{dx} > 0 \Rightarrow V$ is increasing when $0 < x < 18$

$\therefore V$ is has a maximum point at $x = 18$, $l = 108 - 4(18) = 36$

$\therefore$ The dimensions of the largest box are $V = 18 \times 18 \times 36$ inches^3 or $V = 6.75\ ft^3$.

2C-10 A swimmer is on the beach at a point A. The closest point on the straight shoreline to A is called P. There is a platform in the water at B, and the nearest pont on the shoreline to B is called Q. Suppose that the distance from A to P is 100 meters, the distance from B to Q is 100 meters and the distance from P to Q is a meters. Finally suppose that the swimmer can run at 5 meters per second on the beach and swim at 2 meters per second in the water.



Show that the path the swimmer should take to get to the platform in the least time has the property that the ratio of the sines of the angles the path makes with the shoreline is the reciprocal of the ratio of the speeds in the two regions:

$$\frac{\sin \vec{a}}{\sin \vec{b}} = \frac{5}{2}$$

(In optics, this is known as Snell's law describing the path taken by a light ray through two successive media. Snell discovered experimentally that the above ratio of sines was a constant, not depending on the starting point and endpoint of the path. This problem shows that Snell's law follows from a minimum principle: the light ray takes the path minimizing its total travel time. The ratio of sines is constant since it depends only on the speeds of light in the two media.)

$$T = \frac{h_1}{5} + \frac{h_2}{2}$$

7/8/25

$$h_1^2 = x^2 + 100^2$$

$$h_2^2 = (a-x)^2 + 100^2$$

$$\frac{dT}{dx} = \frac{1}{5}\frac{x}{h_1} - \frac{1}{2}\frac{a-x}{h_2}$$

$$2h_1 \frac{dh_1}{dx} = 2x + 0$$

$$\frac{dh_1}{dx} = \frac{x}{h_1}$$

$$2h_2 \frac{dh_2}{dx} = 2(a-x)(-1)$$

$$\frac{dh_2}{dx} = \frac{-a+x}{h_2}$$

At $\frac{dT}{dx} = 0$, $\frac{x}{5h_1} - \frac{a-x}{2h_2} = 0$

$$\frac{a-x}{2h_2} = \frac{x}{5h_1}$$

$$5h_1(a-x) = 2h_2 x$$

$$\frac{5}{2}\frac{(a-x)}{h_2} = \frac{x}{h_1}$$

$$\Rightarrow \frac{5}{2}\sin\beta = \sin\alpha$$

$$\therefore \frac{\sin\alpha}{\sin\beta} = \frac{5}{2}$$

2C-13 a) An airline will fill 100 seats of its aircraft at a fare of \$200. For every \$5 increase in the fare, the plane loses two passengers. For every decrease of \$5, the company gains two passengers. What price maximizes revenue?

b) A utility company has a small power plant that can produce x kilowatt hours of electricity daily at a cost of $10 - x/10^5$ cents each for $0 \le x \le 8 \times 10^5$. Consumers will use $10^5(10 - p/2)$ kilowatt hours of electricity daily at a price of p cents per kilowatt hour. What price should the utility charge to maximize its profit?

5/8/25

a) $R = fp$

$f(i) = 200 + 5i$, $p(i) = 100 - 2i$

$$R = (200+5i)(100-2i)$$
$$= 20000 + 500i - 400i - 10i^2$$
$$= -10(i^2 - 10i - 2000)$$
$$= -10(i-50)(i+40)$$

$$\frac{dR}{di} = -10(2i - 10)$$

$$\frac{dR}{di} = 0 \Rightarrow -10(2i-10) = 0$$
$$i = 5$$

If $i > 5$, $\frac{dR}{di} < 0 \Rightarrow R$ is decreasing,

if $i < 5$, $\frac{dR}{di} > 0 \Rightarrow R$ is increasing

As $i \to \pm\infty$, $R \to -\infty$.

$\therefore$ Revenue is at a maximum at $i = 5$, $f(i) = 225$.

b) $$\text{Cost} = \left(10 - \frac{x}{10^5}\right) \text{cents/kWh} \times x \text{ kWh}$$

$$= \left(10 - \frac{x}{10^5}\right)x \text{ cents}$$

$$\text{Price} = p \text{ cents/kWh} \times 10^5\left(10 - \frac{p}{2}\right) \text{kWh}$$

$$= 10^5 p\left(10 - \frac{p}{2}\right) \text{cents}$$

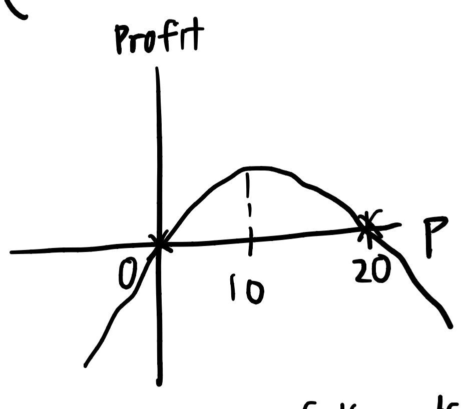
$$\text{Profit} = \text{Price} - \text{Cost}$$

$$= 10^5 p\left(10 - \frac{p}{2}\right) - \left(10 - \frac{x}{10^5}\right)x$$

$$= 10^5 p\left(10 - \frac{p}{2}\right) - \left(10 - \frac{10^5\left(10 - \frac{p}{2}\right)}{10^5}\right)\left(10^5\left(10 - \frac{p}{2}\right)\right)$$

$$= 10^5\left(10 - \frac{p}{2}\right)\left(p - 10 + 10 - \frac{p}{2}\right)$$

$$= 10^5\left(10 - \frac{p}{2}\right)\left(\frac{p}{2}\right) = \frac{10^5}{4}p(20 - p)$$



$$\frac{d}{dp}\text{Profit} = \frac{10^5}{4}\left(1(20 - p) + p(-1)\right)$$

$$= \frac{10^5}{4}(20 - 2p)$$

$$= \frac{10^5}{2}(10 - p)$$

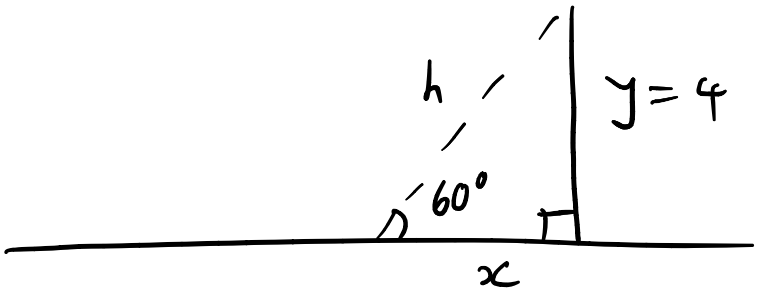
$\therefore$ At price of 10 cents per kWh, profit is maximised.

$$\frac{d}{dp}\text{Profit} = 0 \Rightarrow \frac{10^5}{2}(10 - p) = 0$$

$$p = 10$$

2E-2 A beacon light 4 miles offshore (measured perpendicularly from a straight shoreline) is rotating at 3 revolutions per minute. How fast is the spot of light on the shoreline moving when the beam makes an angle of 60° with the shoreline?

6/8/25



$$w = 3\ r/m = 3(2\pi) = 6\pi$$

$$\tan\theta = \frac{x}{y}$$

$$\Rightarrow \frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{x}{4}$$

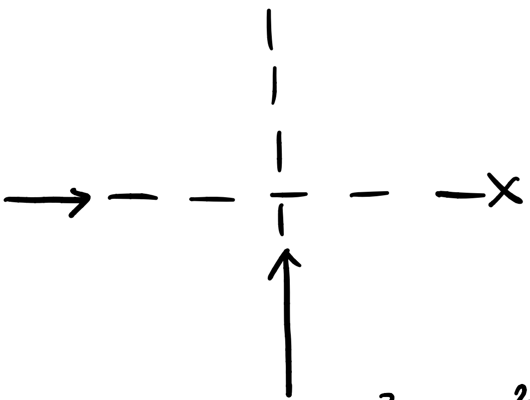
$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{4}\frac{dx}{dt}$$

$$\frac{4}{3}(6\pi) = \frac{1}{4}\frac{dx}{dt}$$

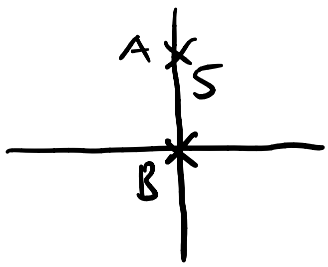
$$\therefore \frac{dx}{dt} = 32\pi \text{ miles per minute}$$

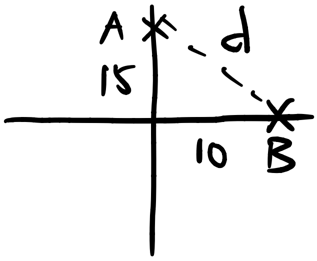
2E-3 Two boats are travelling at 30 miles/hr, the first going north and the second going east. The second crosses the path of the first 10 minutes after the first one was there. At what rate is their distance increasing when the second has gone 10 miles beyond the crossing point?

4/8/25



30 miles per hour
= 0.5 miles per minute





$$d^2 = x^2 + y^2 \Rightarrow d = \sqrt{10^2+15^2}$$
$$= \sqrt{100+225}$$
$$= \sqrt{325}$$
$$= 5\sqrt{13}$$

$$\frac{d}{dt} d^2 = \frac{d}{dt}(x^2+y^2)$$

$$2d\frac{dd}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\Rightarrow \frac{dd}{dt} = \frac{x}{d}\frac{dx}{dt} + \frac{y}{d}\frac{dy}{dt}$$

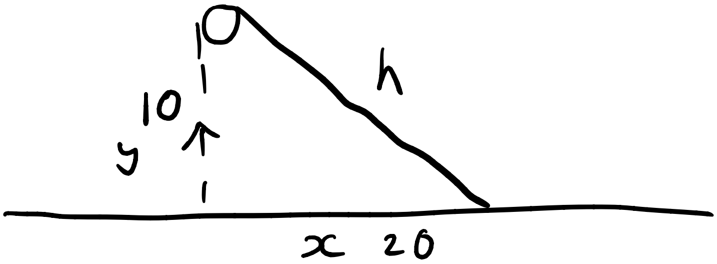
$$= \frac{10}{5\sqrt{13}}30 + \frac{15}{5\sqrt{13}}30$$

$$= \frac{60+90}{\sqrt{13}}$$

$$= \frac{150}{\sqrt{13}}$$

2E-5 A person walks away from a pulley pulling a rope slung over it. The rope is being held at a height 10 feet below the pulley. Suppose that the weight at the opposite end of the rope is rising at 4 feet per second. At what rate is the person walking when s/he is 20 feet from being directly under the pulley?

4/8/25



$$\frac{dh}{dt} = 4 \text{ ft/s}$$

$$h = \sqrt{500} = 10\sqrt{5}$$

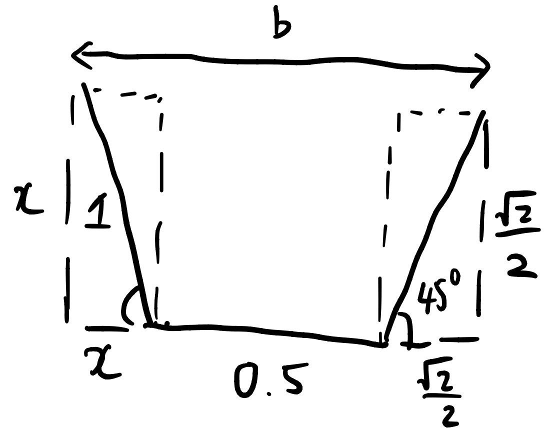
$$x^2 + y^2 = h^2$$

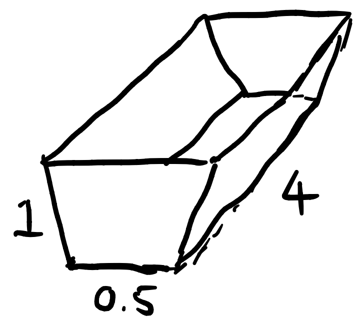
$$\Rightarrow 2x\frac{dx}{dt} + 0 = 2h\frac{dh}{dt}$$

$$\therefore \frac{dx}{dt} = \frac{h}{x}\frac{dh}{dt} = \frac{10\sqrt{5}}{20}4 = 2\sqrt{5} \text{ ft/s}$$

2E-7 A trough is filled with water at a rate of 1 cubic meter per second. The trough has a trapezoidal cross section with the lower base of length half a meter and one meter sides opening outwards at an angle of 45° from the base. The length of the trough is 4 meters. What is the rate at which the water level h is rising when h is one half meter?

5/8/25





$$\frac{dV}{dt} = 1\ cm^3/s$$

$$A = \left(\frac{b+0.5}{2}\right)x$$

$$= \left(\frac{0.5+2x+0.5}{2}\right)x$$

$$= \frac{1}{2}x + x^2$$

$$V = A\ell$$

$$= \left(\frac{x}{2} + x^2\right)4$$

$$= 2x + 4x^2$$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

$$1 = (8x+2)\frac{dx}{dt}$$

$$\therefore \frac{dx}{dt} = \frac{1}{8\left(\frac{1}{2}\right)+2}$$

$$= \frac{1}{6}$$

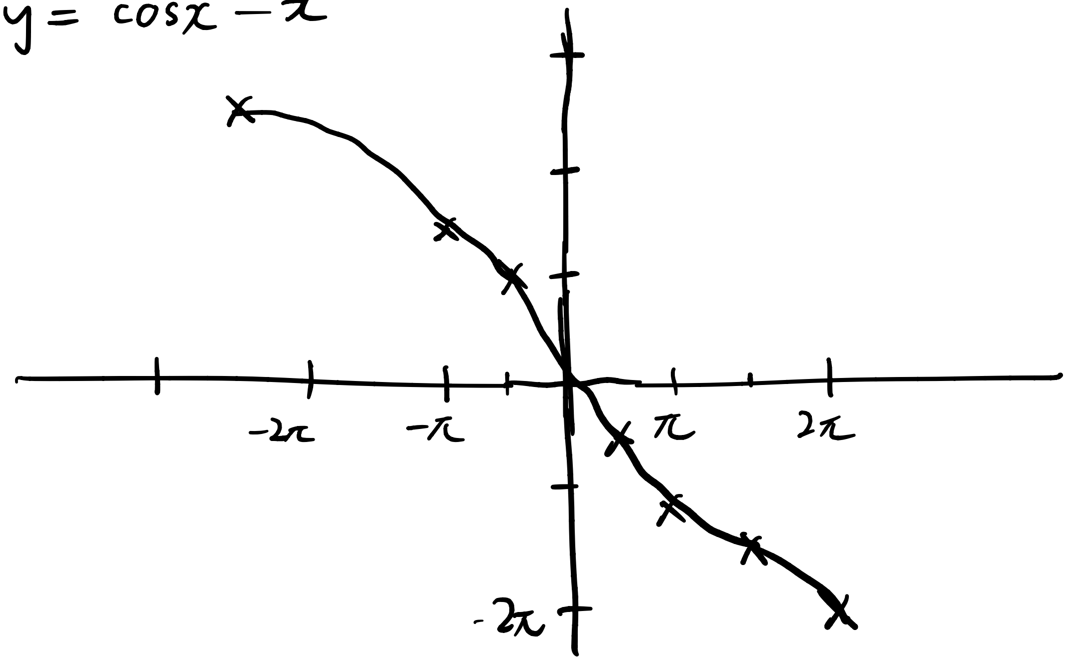
2F-1 a) Graph the function $y = \cos x - x$. Show using y' that there is exactly one root to the equation $\cos x = x$, and give upper and lower bounds on the root.

b) Use Newton's method to find the root to 3 decimal places.

c) Another way to find the root of $\cos x = x$ is to use what is called the fixed point method. Starting with the value $z_1 = 1$, press the cosine key on your calculator until the answer stabilizes, i.e., until $z_{n+1} = \cos z_n$. How many iterations do you need until the first nine digits stabilize? Which method takes fewer steps?

7/8/25

a) $y = \cos x - x$



As $x \to \infty$, $y \to -\infty$

$x \to -\infty$, $y \to \infty$.

$$\frac{dy}{dx} = -\sin x - 1$$

At $\frac{dy}{dx} = 0$, $-\sin x - 1 = 0$

$\sin x = -1$

$$x = -\frac{\pi}{2} + 2k\pi$$

where $k \in \mathbb{Z}$

$\frac{d^2y}{dx^2} = -\cos x \Rightarrow$ concave up when $\frac{\pi}{2} + 2k\pi < x < \frac{3\pi}{2} + 2k\pi$

concave down when $-\frac{\pi}{2} + 2k\pi < x < \frac{\pi}{2} + 2k\pi$

At $y=0$, the graph crosses the x-axis only once.

$\Rightarrow$ Bounds of root: $0 < x < \frac{\pi}{2}$.

b) $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$

$f(0) = \cos 0 - 0 = 1$

$f'(0) = -\sin 0 - 1 = -1$

$x_1 = 0 - \dfrac{1}{-1}$

$= 1$

$f(1) = \cos 1 - 1 = -0.459698$

$f'(1) = -\sin 1 - 1 = -1.841471$

$x_2 = 1 - 0.249636$

$= 0.750364$

$f(0.7503) = -0.018923$

$f'(0.7503) = -1.681905$

$x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$= 0.7391129$

$f(x_3) = -4.6455899 \times 10^{-5}$

$f'(x_3) = -1.673633$

$x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)}$

$= 0.739085$

$f(x_4) = -2.847 \times 10^{-10}$

$f'(x_4) = -1.673612$

$x_5 = 0.739085$

$\therefore x = 0.739$ at $y = 0$.

c) 53 iterations, $x = 0.739085133$

$\therefore$ Newton's method takes fewer steps.